Mathematics Notes for Class 12 chapter 5. Continuity and Differentiability

Derivative

The rate of change of a quantity y with respect to another quantity x is called the derivative or differential coefficient of y with respect to x .

Differentiation of a Function

Let f(x) is a function differentiable in an interval [a, b]. That is, at every point of the interval, the derivative of the function exists finitely and is unique. Hence, we may define a new function g: [a, b] → R, such that, ∀ x ∈ [a, b], g(x) = f'(x).

This new function is said to be differentiation (differential coefficient) of the function f(x) with respect to x and it is denoted by df(x) / d(x) or Df(x) or f'(x).

$$f'(x) = \frac{d}{dx} f(x) = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

Differentiation ‘from First Principle

Let f(x) is a function finitely differentiable at every point on the real number line. Then, its derivative is given by

$$f'(x) = \frac{d}{dx} f(x) = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

Standard Differentiations

1. d / d(x) $(x^n) = nx^{n-1}$, x ∈ R, n ∈ R

2. d / d(x) (k) = 0, where k is constant.

3. d / d(x) $(e^x) = e^x$

4. d / d(x) $(a^x) = a^x \log_e a > 0, a \neq 1$

5. $\frac{d}{dx}(\log_e x) = \frac{1}{x}, x > 0$

6. $\frac{d}{dx}(\log_a x) = \frac{1}{x}(\log_a e) = \frac{1}{x \log_e a}$

7. $\frac{d}{dx}(\sin x) = \cos x$

8. $\frac{d}{dx}(\cos x) = -\sin x$

9. $\frac{d}{dx}(\tan x) = \sec^2 x, x \neq (2n+1)\frac{\pi}{2}, n \in I$

10. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x, x \neq n\pi, n \in I$

11. $\frac{d}{dx}(\sec x) = \sec x \tan x, x \neq (2n+1)\frac{\pi}{2}, n \in I$

12. $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x, x \neq n\pi, n \in I$

13. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, -1 < x < 1$

14. $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}, -1 < x < 1$

15. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

16. $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

17. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}, |x| > 1$

18. $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = -\frac{1}{|x|\sqrt{x^2-1}}, |x| > 1$

19. $\frac{d}{dx}(\sinh x) = \cosh x$

20. $\frac{d}{dx}(\cosh x) = \sinh x$

21. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

Fundamental Rules for Differentiation

(i) $\frac{d}{dx}\{cf(x)\} = c\frac{d}{dx}f(x)$, where c is a constant.

(ii) $\frac{d}{dx}\{f(x) \pm g(x)\} = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$ (sum and difference rule)

(iii) $\frac{d}{dx}\{f(x)\,g(x)\} = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$ (product rule)

Generalization If $u_1, u_2, u_3, \dots, u_n$ be a function of x, then

$$\frac{d}{dx}(u_1\, u_2\, u_3 \dots u_n) = \left(\frac{du_1}{dx}\right)[u_2 u_3 \dots u_n] + u_1\left(\frac{du_2}{dx}\right)[u_3 \dots u_n] + u_1 u_2\left(\frac{du_3}{dx}\right)[u_4 u_5 \dots u_n] + \dots + [u_1 u_2 \dots u_{n-1}]\left(\frac{du_n}{dx}\right)$$

(iv) $\frac{d}{dx}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{\{g(x)^2\}}$ (quotient rule)

(v) if d / d(x) f(x) = φ(x), then d / d(x) f(ax + b) = a φ(ax + b)

(vi) Differentiation of a constant function is zero i.e., d / d(x) (c) = 0.

Geometrically Meaning of Derivative at a Point

Geometrically derivative of a function at a point x = c is the slope of the tangent to the curve y = f(x) at the point {c, f(c)}.

Slope of tangent at P = $\lim_{x \to c}$ f(x) – f(c) / x – c = $\{df(x) / d(x)\}_{x = c}$ or f' (c).

Different Types of Differentiable Function

1. **Differentiation of Composite Function** (Chain Rule)

If f and g are differentiable functions in their domain, then fog is also differentiable and

(fog)' (x) = f' {g(x)} g' (x)

More easily, if y = f(u) and u = g(x), then dy / dx = dy / du * du / dx.

If y is a function of u, u is a function of v and v is a function of x. Then,

dy / dx = dy / du * du / dv * dv / dx.

2. **Differentiation Using Substitution**

In order to find differential coefficients of complicated expression involving inverse trigonometric functions some substitutions are very helpful, which are listed below .

S. No.	Function	Substitution
(i)	$\sqrt{a^2 - x^2}$	$x = a \sin\theta$ or $a \cos\theta$
(ii)	$\sqrt{a^2 + x^2}$	$x = a \tan\theta$ or $a \cot\theta$
(iii)	$\sqrt{x^2 - a^2}$	$x = a \sec\theta$ or $a \operatorname{cosec}\theta$
(iv)	$\sqrt{a + x}$ and $\sqrt{a - x}$	$x = a \cos 2\theta$
(v)	$a \sin x + b \cos x$	$a = r \cos\alpha$, $b = r \sin\alpha$
(vi)	$\sqrt{x - \alpha}$ and $\sqrt{\beta - x}$	$x = \alpha \sin^2\theta + \beta \cos^2\theta$
(vii)	$\sqrt{2ax - x^2}$	$x = a(1 - \cos\theta)$

3. **Differentiation of Implicit Functions**

If f(x, y) = 0, differentiate with respect to x and collect the terms containing dy / dx at one side and find dy / dx.

Shortcut for Implicit Functions For Implicit function, put d /dx {f(x, y)} = – ∂f / ∂x / ∂f / ∂y, where ∂f / ∂x is a partial differential of given function with respect to x and ∂f / ∂y means Partial differential of given function with respect to y.

4. **Differentiation of Parametric Functions**

If x = f(t), y = g(t), where t is parameter, then

dy / dx = (dy / dt) / (dx / dt) = d / dt g(t) / d / dt f(t) = g’ (t) / f’ (t)

5. **Differential Coefficient Using Inverse Trigonometrical Substitutions**

Sometimes the given function can be deducted with the help of inverse Trigonometrical substitution and then to find the differential coefficient is very easy.

(i) $2\sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2})$

(ii) $2\cos^{-1}x = \cos^{-1}(2x^2-1)$ or $\cos^{-1}(1-2x^2)$

(iii) $2\tan^{-1}x = \begin{cases} \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) \\ \tan^{-1}\left(\dfrac{2x-x^2}{1}\right) \\ \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) \end{cases}$

(iv) $3\sin^{-1}x = \sin^{-1}(3x-4x^3)$

(v) $3\cos^{-1}x = \cos^{-1}(4x^3-3x)$

(vi) $3\tan^{-1}x = \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right)$

(vii) $\cos^{-1}x + \sin^{-1}x = \pi/2$

(viii) $\tan^{-1}x + \cot^{-1}x = \pi/2$

(ix) $\sec^{-1}x + \operatorname{cosec}^{-1}x = \pi/2$

(x) $\sin^{-1}x \pm \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right]$

(xi) $\cos^{-1}x \pm \cos^{-1}y = \cos^{-1}\left[xy \mp \sqrt{(1-x^2)(1-y^2)}\right]$

(xii) $\tan^{-1}x \pm \tan^{-1}y = \tan^{-1}\left[\dfrac{x \pm y}{1 \mp xy}\right]$

Logarithmic Differentiation Function

(i) If a function is the product and quotient of functions such as $y = f_1(x)\, f_2(x)\, f_3(x)\ldots / g_1(x)\, g_2(x)\, g_3(x)\ldots$, we first take algorithm and then differentiate.

(ii) If a function is in the form of exponent of a function over another function such as $[f(x)]^{g(x)}$, we first take logarithm and then differentiate.

Differentiation of a Function with Respect to Another Function

Let y = f(x) and z = g(x), then the differentiation of y with respect to z is

dy / dz = dy / dx / dz / dx = f' (x) / g' (x)

Successive Differentiations

If the function y = f(x) be differentiated with respect to x, then the result dy / dx or f' (x), so obtained is a function of x (may be a constant).

Hence, dy / dx can again be differentiated with respect of x.

The differential coefficient of dy / dx with respect to x is written as d /dx (dy / dx) = d^2y / dx^2 or f' (x). Again, the differential coefficient of d^2y / dx^2 with respect to x is written as

d / dx (d^2y / dx^2) = d^3y / dx^3 or f'"(x)……

Here, dy / dx, d^2y / dx^2, d^3y / dx^3,… are respectively known as first, second, third, … order differential coefficients of y with respect to x. These alternatively denoted by f' (x), f'' (x), f''' (x), … or y_1, y_2, y_3…., respectively.

Note dy / dx = (dy / dθ) / (dx / dθ) but $d^2y / dx^2 \neq (d^2y / d\theta^2) / (d^2x / d\theta^2)$

Leibnitz Theorem

If u and v are functions of x such that their nth derivative exist, then

$$D^n(u \cdot v) = {}^nC_0(D^n u)v + {}^nC_1(D^{n-1}u)(Dv) + {}^nC_2(D^{n-2}u)(D^2v) + {}^nC_3(D^{n-3}u)(D^3v) + \ldots + {}^nC_r D^{n-r}u \cdot D^r v + \ldots + {}^nC_n(D^n v)$$

nth Derivative of Some Functions

(i) $\frac{d^n}{dx^n}[\sin(ax+b)] = a^n \sin\left(\frac{n\pi}{2} + ax + b\right)$

(ii) $\frac{d^n}{dx^n}[\cos(ax+b)] = a^n \cos\left(\frac{n\pi}{2} + ax + b\right)$

(iii) $\frac{d^n}{dx^n}(ax+b)^m = \frac{m!}{(m-n)!}a^n(ax+b)^{m-n}$

(iv) $\frac{d^n}{dx^n}[\log(ax+b)] = \frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$

(v) $\frac{d^n}{dx^n}(e^{ax}) = a^n e^{ax}$

(vi) $\frac{d^n}{dx^n}(a^x) = a^x(\log a)^n$

(vii) (a) $\dfrac{d^n}{dx^n}[e^{ax}\sin(bx+c)] = r^n e^{ax}\sin(bx+c+n\phi)$

(b) $\dfrac{d^n}{dx^n}[e^{ax}\cos(bx+c)] = r^n e^{ax}\cos(bx+c+n\phi)$

where, $r = \sqrt{a^2+b^2}$ and $\phi = \tan^{-1}\left(\dfrac{b}{a}\right)$

Derivatives of Special Types of Functions

(i) If $y = f(x)^{\{f(x)\}^{\cdots\infty}}$, then $\dfrac{dy}{dx} = \dfrac{y^2 f'(x)}{f(x)\{1 - y\log f(x)\}}$

(ii) If $e^{g(y)} - e^{-g(y)} = 2f(x)$, then $\dfrac{dy}{dx} = \dfrac{f'(x)}{g'(y)} \cdot \dfrac{1}{\sqrt{1+\{f(x)\}^2}}$

(iii) If $y = \sqrt{\dfrac{1+g(x)}{1-g(x)}}$, then $\dfrac{dy}{dx} = \dfrac{g'(x)}{[1-g(x)]^2} \cdot \sqrt{\dfrac{1-g(x)}{1+g(x)}}$

(iv) If $y = \sqrt{f(x) + \sqrt{f(x) + \sqrt{f(x) + \ldots\infty}}}$, then $\dfrac{dy}{dx} = \dfrac{f'(x)}{2y-1}$

(v) If $\{f(x)\}^{g(y)} = e^{f(x)-g(y)}$, then $\dfrac{dy}{dx} = \dfrac{f'(x)\log f(x)}{g'(y)\{1+\log f(x)\}^2}$

(vi) If $\{f(x)\}^{g(y)} = \{g(y)\}^{f(x)}$, then

$$\frac{dy}{dx} = \frac{g(y)}{f(x)} \cdot \frac{f'(x)}{g'(y)} \left[\frac{f(x)\log g(y) - g(y)}{g(y)\log f(x) - f(x)}\right]$$

(vii) **Differentiation of a Determinant**

If $y = \begin{vmatrix} p & q & r \\ u & v & w \\ l & m & n \end{vmatrix}$, then

$$\frac{dy}{dx} = \begin{vmatrix} \frac{dp}{dx} & \frac{dq}{dx} & \frac{dr}{dx} \\ u & v & w \\ l & m & n \end{vmatrix} + \begin{vmatrix} p & q & r \\ \frac{du}{dx} & \frac{dv}{dx} & \frac{dw}{dx} \\ l & m & n \end{vmatrix} + \begin{vmatrix} p & q & r \\ u & v & w \\ \frac{dl}{dx} & \frac{dm}{dx} & \frac{dn}{dx} \end{vmatrix}$$

(viii) **Differentiation of Integrable Functions** If $g_1(x)$ and $g_2(x)$ are defined in [a, b], Differentiable at $x \in [a, b]$ and f(t) is continuous for $g_1(a) \le f(t) \le g_2(b)$, then

Thus, $\dfrac{\partial f}{\partial x} = \lim\limits_{h \to 0} \dfrac{f(x+h, y) - f(x, y)}{h}$

and $\dfrac{\partial f}{\partial y} = \lim\limits_{k \to 0} \dfrac{f(x, y+k) - f(x, y)}{k}$

Partial Differentiation

The partial differential coefficient of f(x, y) with respect to x is the ordinary differential coefficient of f(x, y) when y is regarded as a constant. It is a written as ∂f / ∂x or $D_x f$ or f_x.

$$\frac{d}{dx}\int_{g_1(x)}^{g_2(x)} f(t)dt = f[g_2(x)]\frac{d}{dx}[g_2(x)] - f[g_1(x)]\frac{d}{dx}[g_1(x)].$$

e.g., If $z = f(x, y) = x^4 + y^4 + 3xy^2 + x^4y + x + 2y$

Then, ∂z / ∂x or ∂f / ∂x or $f_x = 4x^3 + 3y^2 + 2xy + 1$ (here, y is consider as constant)

∂z / ∂y or ∂f / ∂y or $f_y = 4y^3 + 6xy + x^2 + 2$ (here, x is consider as constant)

Higher Partial Derivatives

Let f(x, y) be a function of two variables such that ∂f / ∂x , ∂f / ∂y both exist.

(i) The partial derivative of ∂f / ∂y w.r.t. 'x' is denoted by $\partial^2 f / \partial x^2$ / or f_{xx}.

(ii) The partial derivative of ∂f / ∂y w.r.t. 'y' is denoted by $\partial^2 f / \partial y^2$ / or f_{yy}.

(iii) The partial derivative of ∂f / ∂x w.r.t. 'y' is denoted by $\partial^2 f / \partial y\, \partial x$ / or f_{xy}.

(iv) The partial derivative of ∂f / ∂x w.r.t. 'x' is denoted by $\partial^2 f / \partial y\, \partial x$ / or f_{yx}.

Note $\partial^2 f / \partial x\, \partial y = \partial^2 f / \partial y\, \partial x$

These four are second order partial derivatives.

Euler's Theorem on Homogeneous Function

If f(x, y) be a homogeneous function in x, y of degree n, then

x (&partf / ∂x) + y (&partf / ∂y) = nf

Deduction Form of Euler's Theorem

If f(x, y) is a homogeneous function in x, y of degree n, then

(i) x $(\partial^2 f / \partial x^2)$ + y $(\partial^2 f / \partial x\, \partial y)$ = (n – 1) &partf / ∂x

(ii) x $(\partial^2 f / \partial y\, \partial x)$ + y $(\partial^2 f / \partial y^2)$ = (n – 1) &partf / ∂y

(iii) $x^2 (\partial^2 f / \partial x^2) + 2xy (\partial^2 f / \partial x\, \partial y) + y^2 (\partial^2 f / \partial y^2) = n(n – 1) f(x, y)$

Important Points to be Remembered

If α is m times repeated root of the equation f(x) = 0, then f(x) can be written as

$f(x) = (x - \alpha)^m g(x)$, where $g(\alpha) \neq 0$.

From the above equation, we can see that

$f(\alpha) = 0, f'(\alpha) = 0, f''(\alpha) = 0, \ldots, f^{(m-1)}, (\alpha) = 0$.

Hence, we have the following proposition

$f(\alpha) = 0, f'(\alpha) = 0, f''(\alpha) = 0, \ldots, f^{(m-1)}, (\alpha) = 0$.

Therefore, α is m times repeated root of the equation f(x) = 0.